ON THE BOXICITY OF THE KNESER GRAPH $K(n, 2)$

(Contributed Talk at CanaDAM 2023)

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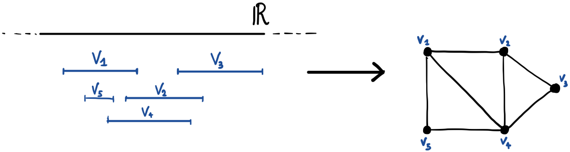
How can we *represent* a graph using *geometric objects*?

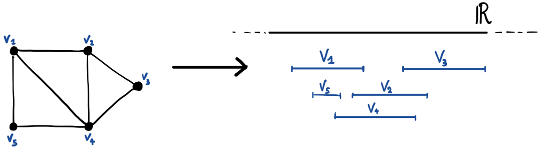
Given a family of (geometric) sets $\mathcal{F}$, the **intersection graph** of $\mathcal{F}$ is the graph $G(\mathcal{F}) = (V_{\mathcal{F}}, E_{\mathcal{F}})$ having:

- $V_{\mathcal{F}} := \{v_A \ : \ A \in \mathcal{F}\}$,
- $E_{\mathcal{F}} := \{v_A v_B \ : \ A, B \in \mathcal{F}, \ A \neq B, \ A \cap B \neq \emptyset\}$.



Roberts' choice: from intervals to boxes

An **interval graph** is a graph that can be represented using a family of closed intervals in the real line.



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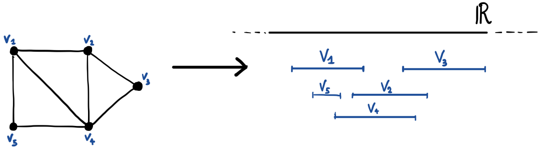
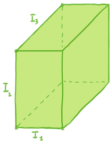


An **axis-parallel box** in $\mathbb{R}^d$ is a Cartesian product $I_1 \times I_2 \times \cdots \times I_d$ where each I_i is a *closed* interval in the real line.



Boxicity

The **boxicity** of a graph $G = (V, E)$, denoted by $\text{box}(G)$, is the minimum dimension d such that G is the intersection graph of a family $(B_v)_{v \in V}$ of d-dimensional boxes in $\mathbb{R}^d$.

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Lemma (Roberts, 1969)

Let $G = (V, E)$ be a simple graph. Then $\text{box}(G) \leq k$ if and only if there are k interval graphs $\{G_i\}_{1 \leq i \leq k}$ on the vertex set $V(G)$ such that:

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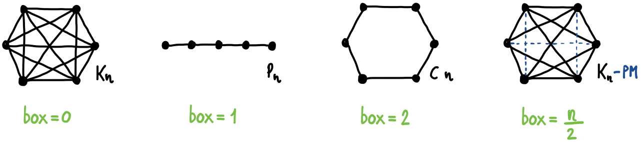

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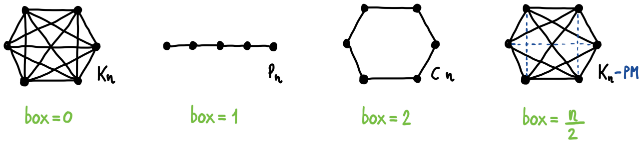

Given k a positive integer and a graph G, deciding if $\text{box}(G) \le k$ is an $\mathcal{NP}$**-complete** problem, even if $k = 2$ [Kratochvíl and Matoušek, 1994].

Motivation

Knowing the boxicity of a graph can be useful:

- It is used to **measure the complexity** of ecological and social networks.
- The geometric representation of a graph can be used to **store** it;
- Several combinatorial problems (e.g., relation between χ and ω) are **studied in details** for graphs of fixed boxicity;

Plan of the talk

- **Overview of main results on boxicity;**
- Our results on the boxicity of the Kneser Graph $K(n, 2)$ (and more...);
- *Key idea*: Relation between interval orders and boxicity;
- Conclusion and open problems.

Bounds using graph parameters

Theorem

Let $G = (V, E)$ be an simple graph on n vertices, m edges, maximum degree Δ, and treewidth tw. Then

- $\text{box}(G) \leq \lfloor \frac{n}{2} \rfloor$ [Roberts, 1969];
- $\text{box}(G) = \mathcal{O}(\sqrt{m \log m})$ [Esperet, 2016];
- $\text{box}(G) = \mathcal{O}(\Delta \log^2 \Delta)$ [Adiga, Bhowmick, and Chandran, 2010];
- $\text{box}(G) \leq tw + 2$ [Chandran and Sivadasan, 2007].

Other bounds involve: *vertex cover number* [Chandran, Das, and Shah, 2007], *degenerecy number* [Adiga, Chandran, and Mathew 2014], etc.

Bounds for special graph classes

Theorem (Esperet and Wiechert, 2018)

Graphs with no K_t-minor have boxicity at most $\mathcal{O}(t^2 \log(t))$.

Theorem (Thomassen, 1986)

Planar graphs have boxicity at most 3. In particular, triangle-free planar graphs have boxicity at most 2.

Other studied classes are: *graphs with bounded genus* [Esperet and Joret, 2013], *bipartite graphs* [Chandran, Das, and Shah, 2007], *line graphs* [Chandran, Mathew, and Sivadasan, 2011], etc.

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Kneser graphs

Let $n, k \in \mathbb{N}$ such that $n \geq 2k$. The **Kneser graph** $\mathrm{K}(n,k) = (V_K, E_K)$ is the graph with:

- $V_K := \{v_A \;:\; A \subseteq [n], |A| = k\}$, where $[n] := \{1, 2, .., n\}$,
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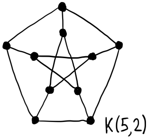

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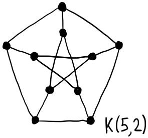

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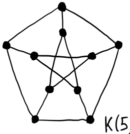

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Theorem (C. and Lichev, 2022)

Fix two positive integers k, n *with* $n \geq 2k$. *The boxicity of the Kneser graph* $\mathsf{K}(n,k)$ *is at most* $n-2$. *Moreover,*

- *if* $n \geq 2k^3 - 2k^2 + 1$, *then* $\mathrm{box}(\mathsf{K}(n,k)) \geq n - \dfrac{13k^2 - 11k + 16}{2}$.
- *if* $k = 2$, *then* $n - 3 \leq \mathrm{box}(\mathsf{K}(n,2)) \leq \mathbf{n-2}$.

Results on complements of line graphs

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Let $G = (V, E)$ *be a graph on* n *vertices and* k *a positive integer. There is an* $\mathcal{O}(n^{5k+3})$*-time algorithm deciding if* $\mathrm{box}(\overline{L(G)}) \leq k$.

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Given a graph $G = (V, E)$, a family $\mathcal{C} = \{C_1, ..C_k\}$ ($C_i \subseteq E$) is a **k-interval-order-cover** of E (or G) if:

- $\bigcup_{i=1}^{k} C_i = E$, and
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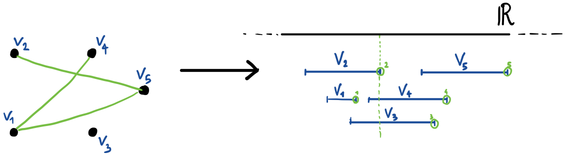
Intuition:
In order to determine the boxicity of $\overline{L(K_n)}$, we need to understand the maximal interval-order subgraphs of $L(K_n)$.

Ordering the vertices

Let $G = (V, E)$ be an interval-order graph on n vertices and $\mathcal{I}$ an interval representation. We can order the interval in $\mathcal{I}$ (and consequently, the vertices in V) in the non-decreasing order $\sigma = V_1 \dots V_n$ of their right end-points.



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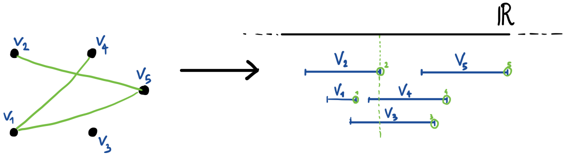

The order of the vertices has the following property:

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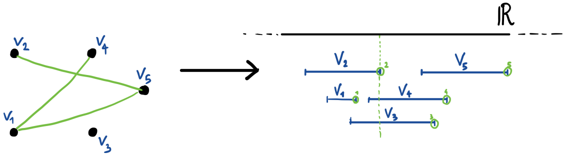

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Lemma (Olariu, 1991)

Let G be a simple graph. Then $G = (V, E)$ is an interval-order graph if and only if V has an order $(v_1, \dots, v_n)$ so that (1) holds.

Subgraph induced by an order

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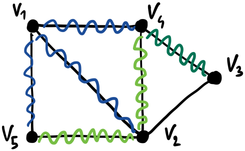
We define $G^\sigma = (V, E^\sigma)$, the **subgraph of G induced by σ**, as follows:

- $V_0 := V$ and $V_i := V_{i-1} \cap N_G(v_i)$ for $1 \leq i \leq n$,
- $E^\sigma := E_1 \cup E_2 \cup \ldots \cup E_{n-1} \subseteq E$, where E_i is the set of edges from v_i to V_i.

$\sigma = (v_1, v_2, v_3, v_4, v_5)$



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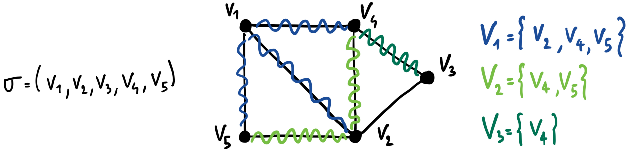

Corollary

For any order σ of V, G^σ is an interval-order subgraph of G. Conversely, any inclusion-wise maximal interval-order subgraph of G is G^σ for some order σ of V.

Lemma (Key Lemma)

Let $n \geq 5$. Then any maximal interval-order subgraph of $L(K_n)$ is a subgraph of $L(K_n)$ of the form defined in (a), (b), or (c) for five different vertices $a, b, c, d, e \in V(K_n)$.

(a) $H = (V, E_{a,b,c,d,e})$,

$$E_{a,b,c,d,e} = Q_a \cup \delta_{ab} \cup \delta_{ac} \cup K_{\{a,b,d\}} \cup K_{\{a,c,e\}}, \ |E_{a,b,c,d,e}| = \frac{\mathbf{(n+2)(n\text{-}1)}}{\mathbf{2}}.$$

(b) $H = (V, E_{a,b,c,d})$,

$$E_{a,b,c,d} = \delta_{ab} \cup \delta_{ad} \cup K_{\{a,b,c,d\}}, \ |E_{a,b,c,d}| = \mathbf{4(n\text{-}1)}.$$

(c) $H = (V, F_{a,b,c,d})$,

$$F_{a,b,c,d} = \delta_{ab} \cup \delta_{ad} \cup \delta_{ac^-} \cup K_{\{a,b,c\}} \cup K_{\{a,b,d\}} \cup K_{d,a,c^-} \cup K_{c,b,d^-},$$

or the same edge set replacing δ_{ad} by δ_{bc}, $|F_{a,b,c,d}| = \mathbf{5(n\text{-}2)}$.

Idea of the proof: First vertex

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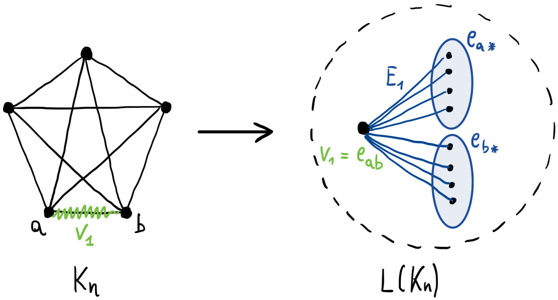

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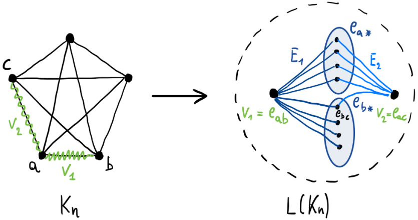

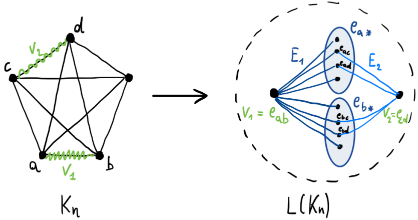

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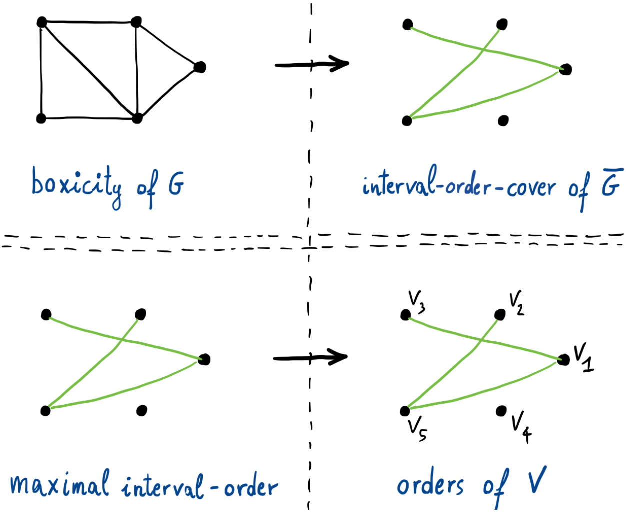
Summary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maximal interval-order
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v_3
v_2
v_1
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Thank you for your attention!